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IDENTITIES ON THE CHANGHEE NUMBERS AND APOSTOL-TYPE DAEHEE POLYNOMIALS

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ABSTRACT. By using generating functions and p-adic integral methods, we study, survey, and investigate various properties of the special numbers and polynomials including the Bernoulli numbers and polynomials, the Euler numbers and polynomials, the Changhee numbers and polynomials, the Daehee numbers and polynomials, the Bernoulli numbers and polynomials of the second kind, the Stirling numbers, and the Catalan numbers. We define Apostol-type Daehee numbers and polynomials of higher order. We derive some properties, relations and identities on these numbers and polynomials. Finally, we give some applications of the p-adic Volkenborn integral including the special numbers and polynomials. We give some remarks and observations associated with the Bernoulli numbers, the Euler numbers, the Daehee numbers, and the Changhee numbers.

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1. INTRODUCTION

Recently, by using the p-adic integral, combinatorial sums, some families of special numbers and polynomials have been studied by many authors (*cf.* [6]-[22], [32], [29], [31], [34]; see also the references cited in each of these earlier works).There are various methods to investigate families of special numbers and polynomials, which are very important all areas of mathematics and mathematical physics. In order to give our results, we use generating functions and p-adic integral methods. In this paper, we give various formulas, relations, and identities related to the Daehee numbers, the Changhee numbers and polynomials, the Bernoulli numbers and polynomials, and the Stirling numbers. We define Apostol-type Daehee numbers and polynomials of higher-order with their generating functions.

We need the following formulas, relations and generating functions for families of well-known special numbers and polynomials.

This paper is dedicated to Professor Dae San KIM on the occasion of his retirement.

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The Apostol-Bernoulli polynomials $B_n(x;\lambda)$ are defined by means of the following generating function:

$$F_A(t,x;\lambda)=\frac{t}{\lambda e^t-1}e^{tx}=\sum_{n=0}^{\infty}B_n(x;\lambda)\frac{t^n}{n!}$$

where λ is a complex number and ($|t|<2\pi$ when $\lambda=1$ and $|t|<|\log\lambda|$ when $\lambda\neq 1$) with

$$B_n(\lambda)=B_n(0;\lambda),$$

which denotes the Apostol-Bernoulli numbers,

$$B_n=B_n(0;1)$$

which denotes the Bernoulli numbers (of the first kind) (*cf.* [17], [8], [23], [24], [25], [26], [29], [33], [34]; see also the references cited in each of these earlier works).

The λ-Stirling numbers of the second kind are defined by means of the following generating function:

$$F_{LS}(t,v;\lambda)=\frac{\left(\lambda e^t-1\right)^v}{v!}=\sum_{n=0}^{\infty}S_2(n,v;\lambda)\frac{t^n}{n!}, \tag{1}$$

(*cf.* [23]; see also [24], [30], [33]; see also the references cited in each of these earlier works).

In [4], the Stirling number of the second kind $S_2(n,k)$ are defined in combinatorics aspect: the Stirling numbers of the second kind are the number of ways to partition a set of n objects into k groups. These numbers are defined by means of the following generating function:

$$F_S(t,v)=\frac{\left(e^t-1\right)^v}{v!}=\sum_{n=0}^{\infty}S_2(n,v)\frac{t^n}{n!}, \tag{2}$$

(*cf.* [1]-[35]; see also the references cited in each of these earlier works). By using the above generating function, these numbers are computed by the following explicit formula:

$$S_2(n,v)=\frac{1}{v!}\sum_{j=0}^{v}\left(\begin{array}{c}v\\ j\end{array}\right)(-1)^j\,(v-j)^n\,. \tag{3}$$

Setting $\lambda=1$ in (1), we have

$$S_2(n,v;1)=S_2(n,v)$$

(*cf.* [1]-[35]; see also the references cited in each of these earlier works).

The Stirling numbers of the first kind $s_1(n,v)$ are defined by means of the following generating function:

$$F_{s1}(t,k)=\frac{(\log(1+t))^k}{k!}=\sum_{n=0}^{\infty}s_1(n,k)\frac{t^n}{n!}, \tag{4}$$

(*cf.* [3], [4], [11], [16], [28], [30], [35]; see also the references cited in each of these earlier works).

The Bernoulli polynomials of the second kind, $b_n(x)$ are defined by means of the following generating function:

$$F_{b2}(t,x)=\frac{t}{\log(1+t)}(1+t)^x=\sum_{n=0}^{\infty}b_n(x)\frac{t^n}{n!} \tag{5}$$

(*cf.* [28]; see also the references cited in therein).

The Bernoulli numbers of the second kind, $b_n(0)$ are defined by means of the following generating function:

$$F_{b2}(t)=\frac{t}{\log(1+t)}=\sum_{n=0}^{\infty}b_n(0)\frac{t^n}{n!}.$$

The numbers $b_n(0)$ are known as the *Cauchy numbers* ([10], [28]).

The Daehee polynomials are defined by means of the following generating function:

$$F_D(t,x)=\frac{\log(1+t)}{t}(1+t)^x=\sum_{n=0}^{\infty}D_n(x)\frac{t^n}{n!}, \tag{6}$$

with

$$D_n=D_n(0)$$

denotes the so-called the *Daehee numbers* (*cf.* [6], [14], [11], [13], [9], [21], [27]; see also the references cited in each of these earlier works).

The Changhee polynomials are defined by means of the following generating function:

$$F_C(t,x)=\frac{2}{2+t}(1+t)^x=\sum_{n=0}^{\infty}Ch_n(x)\frac{t^n}{n!}, \tag{7}$$

with

$$Ch_n=Ch_n(0)$$

denotes the so-called *Changhee numbers* (*cf.* [14], [22], [34]; see also the references cited in each of these earlier works).

In [31], we defined the function $\mathcal{D}_n(x;\lambda)$ by means of the following generating function:

$$F_{\mathcal{D}}(t,x;\lambda)=\frac{\log(1+\lambda t)}{\lambda^{x+1}t}(1+\lambda t)^x=\sum_{n=0}^{\infty}\mathcal{D}_n(x;\lambda)\frac{t^n}{n!}, \tag{8}$$

with

$$D_n(\lambda)=\mathcal{D}_n(0;\lambda)$$

denotes the so-called the *Apostol-type Daehee numbers.*

Throughout this paper, we use the following notations:

$\mathbb{N}=\{1,2,3,\ldots\}$, $\mathbb{N}_0=\{0,1,2,3,\ldots\}=\mathbb{N}\cup\{0\}$ and $\mathbb{Z}^-=\{-1,-2,-3,\ldots\}$. Here, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{C}$ denotes the set of complex numbers and $\mathbb{Z}_p$ denotes the set of p-adic integers. We assume that $\ln(z)$ denotes the principal branch of the multi-valued function $\ln(z)$ with the imaginary part $Im(\ln(z))$ constrained by $-\pi<Im(\ln(z))\leq\pi$. Furthermore, $0^n=1$ if $n=0$, and, $0^n=0$ if $n\in\mathbb{N}$.

$$\binom{x}{v}=\frac{x(x-1)\cdots(x-v+1)}{v!}=\frac{(x)_v}{v!}$$

(*cf.* [1]-[35]; see also the references cited in each of these earlier works).

We summarize our results as follows.

In Section 2, we define the Apostol-type Daehee numbers and polynomials of higher order. We investigate some properties of these numbers and polynomials.

In Section 3, we give some identities and relations including the Apostol-type Daehee numbers and polynomials of higher order, the Changhee numbers and polynomials and the Stirling numbers.

In Section 4, we give further remarks and observation on the Changhee and Daehee numbers and polynomials and also the Catalan numbers related to application of the p-adic integrals.

In Section 5, we give some remarks and observations related to the Bernoulli numbers, the Euler numbers, the Daehee numbers, and the Changhee numbers.

2. Apostol-type Daehee numbers and polynomials of higher order

In this section, we define Apostol-type Daehee numbers and polynomials of higher order. We give some properties of these numbers and polynomials. The functions, $\mathcal{D}_n^{(k)}(x;\lambda)$ are defined by means of the following generating functions:

$$F_{\mathcal{D}}(t,x;\lambda,k)=\left(\frac{\log(1+\lambda t)}{\lambda^{x+1}t}\right)^k(1+\lambda t)^x=\sum_{n=0}^{\infty}\mathcal{D}_n^{(k)}(x;\lambda)\frac{t^n}{n!}. \tag{9}$$

Note that there is one generating function for each nonnegative value of k.

Observe that substituting $\lambda=1$ into (9), $\mathcal{D}_n^{(k)}(x;1)$ reduces to the higher-order Daehee Polynomials (*cf.* [5], [15]). These numbers, $D_n^{(k)}(\lambda)$ are defined by means of the following generating functions:

$$F_D(t;\lambda,k)=\left(\frac{\log(1+\lambda t)}{\lambda t}\right)^k=\sum_{n=0}^{\infty}D_n^{(k)}(\lambda)\frac{t^n}{n!}. \tag{10a}$$

Note that there is one generating function for each value of k.

We give the following functional equation:

$$F_D(t;\lambda,a+b)=F_D(t;\lambda,a)F_D(t;\lambda,b) \tag{11}$$

and

$$F_{\mathcal{D}}(t,x;\lambda,a+b)=F_{\mathcal{D}}(t,x;\lambda,a)F_{\mathcal{D}}(t,x;\lambda,b). \tag{12}$$

Combining (11) with (10a), we obtain

$$\sum_{n=0}^{\infty}D_n^{(a+b)}(\lambda)\frac{t^n}{n!}=\sum_{n=0}^{\infty}D_n^{(a)}(\lambda)\frac{t^n}{n!}\sum_{n=0}^{\infty}D_n^{(b)}(\lambda)\frac{t^n}{n!}.$$

Therefore,

$$\sum_{n=0}^{\infty}D_n^{(a+b)}(\lambda)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\binom{n}{j}\mathcal{D}_j^{(a)}(x;\lambda)\mathcal{D}_{n-j}^{(b)}(x;\lambda)\right)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the following theorem:

Theorem 2.1.

$$D_n^{(a+b)}(\lambda) = \sum_{j=0}^{n} \binom{n}{j} D_j^{(a)}(\lambda) D_{n-j}^{(b)}(\lambda). \tag{13}$$

Theorem 2.2.

$$\mathcal{D}_n^{(a+b)}(x;\lambda) = \sum_{j=0}^{n} \binom{n}{j} \mathcal{D}_j^{(a)}(x;\lambda) \mathcal{D}_{n-j}^{(b)}(x;\lambda). \tag{14}$$

The proof of Equation (14) can be shown by using (12) and (9). This proof is same as that of (13). Therefore, the details of the proof is omitted.

We take derivative of (9) with respect to x, we obtain the following a partial differential equation:

$$\frac{\partial}{\partial x} F_{\mathcal{D}}(t,x;\lambda,k) = kF_{\mathcal{D}}(t,x;\lambda,k)\left(tF_D(t,0;\lambda,1) - \log\lambda\right).$$

Combining the above equation with (9), we have

$$\sum_{n=0}^{\infty} \frac{\partial}{\partial x} \mathcal{D}_n^{(k)}(x;\lambda)\frac{t^n}{n!} = k\sum_{n=0}^{\infty} \mathcal{D}_n^{(k)}(x;\lambda)\frac{t^n}{n!}\left(\sum_{n=0}^{\infty} \mathcal{D}_n(\lambda)\frac{t^{n+1}}{n!} - \log\lambda\right).$$

By using the above equation, we obtain derivative formula for the $\mathcal{D}_n^{(k)}(x;\lambda)$ by the following theorem:

Theorem 2.3.

$$\frac{\partial}{\partial x}\mathcal{D}_n^{(k)}(x;\lambda) = nk\sum_{j=0}^{n-1} \binom{n-1}{j} \mathcal{D}_j^{(k)}(x;\lambda) D_{n-1-j}(\lambda) - k\left(\log\lambda\right)\mathcal{D}_n^{(k)}(x;\lambda).$$

We now modify generating functions in (9). We define *Apostol-type Daehee polynomials* $\mathfrak{D}_n^{(k)}(x;\lambda)$ of higher-order by means of the following generating function:

(15)

$$G_{\mathfrak{D}}(t,x;\lambda,k) = \left(\frac{\log(\lambda) + \log(1+\lambda t)}{\lambda^2 t + \lambda - 1}\right)^k (1+\lambda t)^x = \sum_{n=0}^{\infty} \mathfrak{D}_n^{(k)}(x;\lambda)\frac{t^n}{n!}.$$

Substituting $x = 0$ into (15), we obtain the following *Apostol-type Daehee numbers* $\mathfrak{D}_n^{(k)}(\lambda)$ of higher-order:

$$\mathfrak{D}_n^{(k)}(\lambda) = \mathfrak{D}_n^{(k)}(0;\lambda).$$

Remark 1. *Substituting $\lambda = 1$ into (15), $\mathfrak{D}_n^{(k)}(x;\lambda)$ reduces to the Daehee polynomials $D_n^{(k)}(x,\lambda)$ of higher-order (cf.* [15]*; see also the references cited in each of these earlier works).*

Remark 2. *El-Desouky and Mustafa* [5]*, defined so-called the λ-Daehee polynomials of the first kind with order k by the generating function:*

$$\left(\frac{\lambda\log(1+\lambda t)}{(t+1)^{\lambda} - 1}\right)^k (1+\lambda t)^x = \sum_{n=0}^{\infty} D_{n,\lambda}^{(k)}(x)\frac{t^n}{n!}.$$

Substituting

$$\lambda t = e^z - 1$$

into (15), we get

$$\left(\frac{\log(\lambda) + z}{\lambda e^z - 1}\right)^k e^{zx} = \sum_{n=0}^{\infty} \mathfrak{D}_n^{(k)}(x;\lambda) \frac{(e^z - 1)^n}{\lambda^n n!}.$$

Combining right-hand side of the above equation with (2), we obtain

$$\left(\frac{\log(\lambda) + z}{\lambda e^z - 1}\right)^k e^{zx} = \sum_{n=0}^{\infty} \mathfrak{D}_n^{(k)}(x;\lambda) \frac{1}{\lambda^n} \sum_{m=0}^{\infty} S_2(m,n) \frac{z^m}{m!}.$$

Since $S_2(m,n) = 0$ when $m < n$, we have

$$\left(\frac{\log(\lambda) + z}{\lambda e^z - 1}\right)^k e^{zx} = \sum_{m=0}^{\infty} \sum_{n=0}^{m} S_2(m,n) \mathfrak{D}_n^{(k)}(x;\lambda) \frac{1}{\lambda^n} \frac{z^m}{m!}. \tag{16}$$

The left-hand side of the above equation (16) is a generating function for the well-known λ-Bernoulli polynomials $\mathfrak{B}_n(x;\lambda)$, which is defined as follows:

$$F_B(t,x;\lambda,k) = \left(\frac{\log\lambda + t}{\lambda e^t - 1}\right)^k e^{tx} = \sum_{n=0}^{\infty} \mathfrak{B}_n^{(k)}(x;\lambda) \frac{t^n}{n!} \tag{17}$$

(*cf.* [19]). Combining (16) and (17), we have

$$\sum_{m=0}^{\infty} \mathfrak{B}_n^{(k)}(x;\lambda) \frac{z^m}{m!} = \sum_{m=0}^{\infty} \sum_{n=0}^{m} S_2(m,n) \mathfrak{D}_n^{(k)}(x;\lambda) \frac{1}{\lambda^n} \frac{z^m}{m!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the following theorem:

Theorem 2.4.

$$\mathfrak{B}_n^{(k)}(x;\lambda) = \sum_{n=0}^{m} \frac{1}{\lambda^n} S_2(m,n) \mathfrak{D}_n^{(k)}(x;\lambda).$$

Observe that $\mathfrak{D}_n^{(k)}(x;\lambda)$ is a polynomial of variable x. These polynomials are associated with the Apostol-type Bernoulli polynomials, the Frobenius-Euler numbers, and the other special polynomials.

3. Identities and Relations

In this section, we derive some identities and relations including the Apostol-type Daehee numbers and polynomials of higher-order, the Changhee numbers and polynomials and the Stirling numbers by using generating function and their functional equation.

By using (7), we obtain

$$(1+t)^x = \sum_{n=0}^{\infty} \frac{n}{2} Ch_{n-1}(x) \frac{t^n}{n!} + \sum_{n=0}^{\infty} 2Ch_n(x) \frac{t^n}{n!}.$$

From the above equation, we have

$$\sum_{n=0}^{\infty} \left(\frac{n}{2} Ch_{n-1}(x) + Ch_n(x)\right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} (x)_n \frac{t^n}{n!}. \tag{18}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the following theorem:

Theorem 3.1. *(Recurrence relation for the Changhee polynomials)*

$$\frac{n}{2}Ch_{n-1}(x)+Ch_n(x)=(x)_n\,. \tag{19}$$

By (19), we compute few values of the Changhee polynomial as follows:

$$\begin{aligned}Ch_0(x) &= 1,\\ Ch_1(x) &= x-\frac{1}{2},\\ Ch_2(x) &= x^2-2x+\frac{1}{2}\\ Ch_3(x) &= x^3-\frac{9}{2}x^2+5x-\frac{3}{4},\end{aligned}$$

etc.

In [31], we gave the λ-Bernoulli polynomials of the second kind are defined by means of the following generating function:

$$F_{b2}(t,x;\lambda)=\frac{\lambda^{1-x}t}{\log(1+\lambda t)}(1+\lambda t)^x=\sum_{n=0}^{\infty}b_n(x;\lambda)\frac{t^n}{n!}. \tag{20}$$

By using (20) and (8), we have the following functional equation:

$$F_{\mathcal{D}}(t,x)F_{b2}(t,x)=\lambda^{-2x}(1+\lambda t)^{2x}.$$

Combining the above functional equation with (19), we get

$$\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}b_j(x;\lambda)\mathcal{D}_{n-j}(x;\lambda)\frac{t^n}{n!}=\lambda^{-2x}\sum_{n=0}^{\infty}\left(\frac{n}{2}Ch_{n-1}(x)+Ch_n(x)\right)\lambda^n\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we obtain the following theorem:

Theorem 3.2.

$$\sum_{j=0}^{n}\binom{n}{j}b_j(x;\lambda)\mathcal{D}_{n-j}(x;\lambda)=\lambda^{n-2x}\left(\frac{n}{2}Ch_{n-1}(x)+Ch_n(x)\right). \tag{21}$$

Theorem 3.3.

$$D_n^{(k)}(x;\lambda)=\frac{\lambda^{n-k(x+1)}}{\binom{n}{k}}\sum_{j=0}^{n}\binom{n}{j}s_1(n-j,k)\left(\frac{j}{2}Ch_{j-1}(x)+Ch_j(x)\right).$$

Proof. Combining (9) with (4), we get the following functional equation:

$$\lambda^{k(x+1)}t^kF_{\mathcal{D}}(t,x;\lambda,k)=k!F_{s1}(\lambda t,k)(1+\lambda t)^x.$$

By using the above equation, we obtain

$$\lambda^{k(x+1)}\sum_{n=0}^{\infty}\mathcal{D}_n^{(k)}(x;\lambda)\frac{t^{n+k}}{n!}=k!\left(\sum_{n=0}^{\infty}s_1(n,k)\frac{(\lambda t)^n}{n!}\right)(1+\lambda t)^x.$$

We assume that $|\lambda t|<1$, then the above equation reduces to the following equation

$$\lambda^{k(x+1)}\sum_{n=0}^{\infty}\mathcal{D}_n^{(k)}(x;\lambda)\frac{t^{n+k}}{n!}=k!\sum_{n=0}^{\infty}s_1(n,k)\frac{(\lambda t)^n}{n!}\sum_{n=0}^{\infty}(x)_n\frac{(\lambda t)^n}{n!}.$$

Therefore,

$$\lambda^{k(x+1)}\sum_{n=0}^{\infty}(n)_k\mathcal{D}_{n-k}^{(k)}(x;\lambda)\frac{t^n}{n!}=k!\sum_{n=0}^{\infty}\sum_{j=0}^{n}\left(\begin{array}{c}n\\ j\end{array}\right)\lambda^n s_1(n-j,k)(x)_j\frac{t^n}{n!}.$$

Substituting equation (19) into the above equation, we get

$$\begin{aligned}&\lambda^{k(x+1)}\sum_{n=0}^{\infty}(n)_k\mathcal{D}_{n-k}^{(k)}(x;\lambda)\frac{t^n}{n!}\\ =\;& k!\sum_{n=0}^{\infty}\sum_{j=0}^{n}\left(\begin{array}{c}n\\ j\end{array}\right)\lambda^n s_1(n-j,k)\left(\frac{j}{2}Ch_{j-1}(x)+Ch_j(x)\right)\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

We define the following functional equation:

$$G_{\mathfrak{D}}(t,x;\lambda,k)=\frac{(1+\lambda t)^x}{(\lambda-1)^k\left(\frac{\lambda^2}{\lambda-1}t+1\right)^k}\sum_{j=0}^{k}\binom{k}{j}j!\left(\log(\lambda)\right)^{k-j}F_{s1}(\lambda t,j).$$

By combining the above equation with (15) and (4), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\mathfrak{D}_n^{(k)}(x;\lambda)\frac{t^n}{n!}&=\frac{1}{(\lambda-1)^k}\sum_{n=0}^{\infty}(x)_n\frac{(\lambda t)^n}{n!}\sum_{n=0}^{\infty}\binom{n+k-1}{n}\left(-\frac{\lambda^2 t}{\lambda-1}\right)^n\\ &\quad\sum_{j=0}^{k}\binom{k}{j}j!\left(\log(\lambda)\right)^{k-j}\sum_{n=0}^{\infty}s_1(n,j)\frac{(\lambda t)^n}{n!}\end{aligned}$$

By combining the above equation with (18) and (4), we arrive at the following theorem including a relation between the Apostol-type Daehee polynomials and Changhee polynomials:

Theorem 3.4. *Let n be a positive integer. Then*

$$\begin{aligned}\mathfrak{D}_n^{(k)}(x;\lambda)&=\sum_{j=0}^{k}\sum_{c=0}^{n}\sum_{m=0}^{c}(-1)^{n-c}\binom{n}{c}\binom{k}{j}\binom{n+k-c-1}{n-c}\\ &\quad\times\frac{\lambda^{2n+c}n!j!\left(\log\lambda\right)^{k-j}}{(1-\lambda)^{n+k-c}c!}s_1(c-m,j)(x)_m.\end{aligned}\tag{22}$$

Substituting (19) into (22), we arrive at the following corollary:

Corollary 3.5.

$$\begin{aligned}\mathfrak{D}_{n}^{(k)}(x;\lambda) &= \sum_{j=0}^{k}\sum_{c=0}^{n}\sum_{m=0}^{c}(-1)^{n-c}\binom{n}{c}\binom{k}{j}\binom{n+k-c-1}{n-c} \\ &\times\frac{\lambda^{2n+c}n!j!\,(\log\lambda)^{k-j}}{(1-\lambda)^{n+k-c}c!}s_1(c-m,j)\left(\frac{m}{2}Ch_{m-1}(x)+Ch_m(x)\right).\end{aligned}$$

4. Further Remarks and observation on the Changhee and Daehee numbers and polynomials: Applying the p-adic integrals

By using the p-adic integrals, various different properties of the combinatorial sums, special numbers, and polynomials have been studied. In this section, we investigate and survey some properties of the Changhee and Daehee numbers and polynomials by using the p-adic integrals (*cf.* [6]-[22], [32], [29], [31], [34]; see also the references cited therein).

Let p be a fixed prime and $UD(\mathbb{Z}_p)$ be the set of uniformly differentiable functions on $\mathbb{Z}_p$. The p-adic q-integral of the function $f\in UD(\mathbb{Z}_p)$ was defined by Kim [16] as follows:

$$\int_{\mathbb{Z}_p} f(x)d\mu_q(x)=\lim_{N\to\infty}\frac{1}{[p^N]_q}\sum_{x=0}^{p^N-1}f(x)q^x,$$

where

$$[x]=\frac{1-q^x}{1-q}$$

and

$$\mu_q(x)=\mu_q(x+p^N\mathbb{Z}_p)=\frac{q^x}{[p^N]_q},$$

where $q\in\mathbb{C}_p$ with $|\,1-q\,|_p<1$.

The *bosonic* p-adic integral (the Volkenborn integral) is given by

$$\int_{\mathbb{Z}_p} f(x)\,d\mu_1(x)=\lim_{N\to\infty}\frac{1}{p^N}\sum_{x=0}^{p^N-1}f(x),\tag{23}$$

where

$$\mu_1(x)=\lim_{q\to1}\mu_q(x)$$

(*cf.* [32], [16]; see also the references cited therein).

A relation between the Volkenborn integral and the Bernoulli numbers is given by

$$\int_{\mathbb{Z}_p} x^n d\mu_1(x)=B_n\tag{24}$$

(*cf.* [16], [17], [29], [32], [34]; see also the references cited in each of these earlier works).

The so-called the *fermionic* p-adic integral on $\mathbb{Z}_p$ is given by

$$\int_{\mathbb{Z}_p} f(x)\, d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} (-1)^x f(x) \tag{25}$$

where

$$\mu_{-1}(x) = \lim_{q\to -1} \mu_q(x)$$

(*cf.* [17]). By using (25), one has a relation between the fermionic p-adic integral and the Euler numbers E_n as follows:

$$\int_{\mathbb{Z}_p} x^n d\mu_{-1}(x) = E_n, \tag{26}$$

(*cf.* [17], [8], [29], [34]; see also the references cited in each of these earlier works).

Theorem 4.1.

$$\int_{\mathbb{Z}_p} \binom{x}{j} d\mu_1(x) = \frac{(-1)^j}{j+1}. \tag{27}$$

Proof of Theorem 4.1 was given by Schikhof [32].

Theorem 4.2.

$$\int_{\mathbb{Z}_p} \binom{x}{j} d\mu_{-1}(x) = \frac{(-1)^j}{2^j}. \tag{28}$$

Proof of Theorem 4.2 was given by Kim *et al.* [14].

Applying the Volkenborn integral, the Witt-type formula for the Daehee numbers and polynomials were given by Kim *at al.* [14] as follows, respectively:

$$D_n = \int_{\mathbb{Z}_p} (x)_n d\mu_1(x)$$

and

$$D_n(y) = \int_{\mathbb{Z}_p} (y+x)_n d\mu_1(x).$$

Applying the fermionic p-adic integral, the Witt-type formula for the Changhee numbers and polynomials were given by Kim *at al.* [14] as follows, respectively:

$$Ch_n = \int_{\mathbb{Z}_p} (x)_n d\mu_{-1}(x)$$

and

$$Ch_n(y) = \int_{\mathbb{Z}_p} (y+x)_n d\mu_{-1}(x).$$

Remark 3. *In* [18, p. 5 Theorem 3]*, Kim defined the Catalan numbers,* C_n *via the fermionic p-adic integral as follows:*

$$C_n = (-1)^n 2^{2n} \int\limits_{\mathbb{Z}_p} \binom{\frac{x}{2}}{n} d\mu_{-1}(x),$$

where

$$C_n = \frac{1}{n+1} \binom{2n}{n}.$$

From the above relation, we have

$$\int\limits_{\mathbb{Z}_p} \left(\frac{x}{2}\right)_n d\mu_{-1}(x) = (-1)^n \frac{n!}{2^{2n}} C_n.$$

Remark 4. *There are many applications of the fermionic and bosonic p-adic integral special numbers and polynomials, combinatorial sums, and formulas (cf.* [6]-[22], [32], [29], [31], [34]*; see also the references cited therein).*

4.1. **Observation on generating function for the Apostol-type Daehee numbers.** By using the same method that of Kim [15], we give a generating function for the Apostol-type Daehee numbers of higher-order.

We set

$$f(x,t;\lambda) = \lambda^x (1+\lambda t)^x.$$

From the above function, we have

$$f(x+1,t;\lambda) = \lambda (1+\lambda t) f(x,t;\lambda)$$

and

$$\frac{\partial}{\partial t} f(x,t;\lambda)\mid_{x=0} = \log(\lambda) + \log(1+\lambda t).$$

By substituting the above function and relations into the following integral equation of Equation (23), we obtain

$$\log(\lambda)+\log(1+\lambda t) = \lambda(1+\lambda t) \int\limits_{\mathbb{Z}_p} \lambda^x (1+\lambda t)^x d\mu_1(x) - \int\limits_{\mathbb{Z}_p} \lambda^x (1+\lambda t)^x d\mu_1(x).$$

Therefore,

$$\int\limits_{\mathbb{Z}_p} \lambda^x (1+\lambda t)^x d\mu_1(x) = \frac{\log(\lambda) + \log(1+\lambda t)}{\lambda(1+\lambda t) - 1}.$$

By using the above integral equation, we define

$$G(t;\lambda) = \frac{\log(\lambda) + \log(1+\lambda t)}{\lambda(1+\lambda t) - 1}. \tag{29}$$

We investigate some properties of the above function. Substituting

$$t = \frac{e^z - 1}{\lambda}$$

into (29), we obtain

$$G\left(\frac{e^z-1}{\lambda};\lambda\right) = \frac{\log(\lambda) + z}{\lambda e^z - 1}.$$

We observe that $G\left(\frac{e^z-1}{\lambda};\lambda\right)$ is a generating function for the λ-Bernoulli numbers (*cf.* [19]). We also observe that $G\left(e^z-1;1\right)$ is a generating function for the Bernoulli numbers.

Substituting $\lambda=1$ into (29), we obtain

$$G\left(t;1\right)=\frac{\log\left(1+t\right)}{t}.$$

Observe that $G\left(t;1\right)$ is a generating function for the Deahee numbers (*cf.* [14], [15]).

5. Sequences of the Bernoulli numbers and the Euler numbers related to the Changhee and the Daehee numbers

In this section, we study sequences including the Bernoulli numbers and the Euler numbers related to the Changhee numbers and the Daehee numbers. By combining (24) with (27), one has the following relations for the Daehee numbers and the Bernoulli numbers:

$$\begin{aligned}
D_0 &= B_0,\\
D_1 &= B_1,\\
D_2 &= B_2-B_1,\\
D_3 &= B_3-3B_2+2B_1,\\
D_4 &= B_4-6B_3+11B_2-6B_1,\ldots
\end{aligned}$$

That is

$$D_0=1, D_1=-\frac{1}{2}, D_2=\frac{2}{3}, D_3=-\frac{3}{2}, D_4=\frac{24}{5},\cdots$$

Remark 5. *Consequently, in the above computation, we see that the Deahee numbers D_n are associated with a sequence the Bernoulli numbers B_n and the Deahee polynomials $D_n(y)$ are associated with a sequence the Bernoulli polynomials $B_n(y)$.*

By combining (26) with (28), one has the following relations for the Changhee numbers and the Euler numbers:

$$\begin{aligned}
Ch_0 &= E_0,\\
Ch_1 &= E_1,\\
Ch_2 &= E_2-E_1,\\
Ch_3 &= E_3-3E_2+2E_1,\\
Ch_4 &= E_4-6E_3+11E_2-6E_1,
\end{aligned}$$

that is

$$Ch_0=1, Ch_1=-\frac{1}{2}, Ch_2=\frac{1}{2}, Ch_3=-\frac{3}{4}, Ch_4=\frac{3}{2},\ldots$$

Remark 6. *Consequently, similarly, in the above computation, we see that the Changhee numbers Ch_n are associated with a sequence the Euler numbers E_n and the Changhee polynomials $Ch_n(y)$ are associated with a sequence the Euler polynomials $E_n(y)$.*

By spirit of the above computation, we catch up with two explicit forms for the Changhee numbers and the Daehee numbers in [14] and [1], respectively, as follows:

In [14, p. 5972, Equation 2.10], Kim *et al.* gave an explicit formula for the Daehee numbers as follows:

$$D_n = \sum_{l=0}^{n} s_1(n,l)\, B_l.$$

and in [1, p. 996, Theorem 2.5], Kim *et al.* also gave an explicit formula for the Changhee numbers as follows:

$$Ch_n = \sum_{l=0}^{n} s_1(n,l)\, E_l.$$

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